

THE GEOMETRY OF HEMI-SLANT SUBMANIFOLDS OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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ABSTRACT. In the present paper, we study hemi-slant submanifolds of a locally product Riemannian manifold. We prove that the anti-invariant distribution which is involved in the definition of hemi-slant submanifold is integrable and give some applications of this result. We get a necessary and sufficient condition for a proper hemi-slant submanifold to be a hemi-slant product. We also study this type submanifolds with parallel canonical structures. Moreover, we give two characterization theorems for the totally umbilical proper hemi-slant submanifolds. Finally, we obtain a basic inequality involving Ricci curvature and the squared mean curvature of a hemi-slant submanifold of a certain type locally product Riemannian manifold.

1. INTRODUCTION

Study of slant submanifolds was initiated by B.Y. Chen [8], as a generalization of both holomorphic and totally real submanifolds of a Kähler manifold. Slant submanifolds have been studied in different kind structures; almost contact [13], neutral Kähler [4], Lorentzian Sasakian [2] and Sasakian [6] by several geometers. N. Papaghiuc [14] introduced semi-slant submanifolds of a Kähler manifold as a natural generalization of slant submanifold. A. Carriazo [7], introduced bi-slant submanifolds of an almost Hermitian manifold as a generalization of semi-slant submanifolds. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds which are studied by A. Carriazo [7]. However, B. Şahin [18] called these submanifolds as hemi-slant submanifolds because of that the name anti-slant seems to refer that it has no slant factor. We observe that a hemi-slant submanifold is a special case of generic submanifold which was introduced by G.S. Ronsse [16]. Since then many geometers have studied hemi-slant submanifolds in different kind structures; Kähler [3, 18], nearly Kähler [21], generalized complex space form [20] and almost Hermitian [19]. We note that sometimes hemi-slant submanifolds are also studied under the name pseudo-slant submanifolds, see [11] and [21]. The submanifolds of a locally product Riemannian manifold have been studied by many geometers. For example, T. Adati [1] defined and studied invariant and anti-invariant submanifolds, while A. Bejancu [5] and G. Pitis [15] studied semi-invariant submanifolds. Slant and semi-slant submanifolds of a locally product Riemannian manifold are examined by B. Şahin [17] and H. Li and X. Liu [12]. In this paper, we study hemi-slant submanifolds of a locally product Riemannian manifold in detail.

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2. PRELIMINARIES

This section is devoted to preliminaries. Actually, in subsection 2.1 we present the basic background needed for a locally product Riemannian manifold. Theory of submanifolds and distributions related to the study are given in subsection 2.2.

2.1. Locally product Riemannian manifolds. Let \bar{M} be an m -dimensional manifold with a tensor field of type (1,1) such that

$$(2.1) \quad F^2 = I, (F \neq \pm I) ,$$

where I is the identity morphism on the tangent bundle $T\bar{M}$ of \bar{M} . Then we say that \bar{M} is an *almost product manifold* with almost product structure F . If an almost product manifold (\bar{M}, F) admits a Riemannian metric g such that

$$(2.2) \quad g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V})$$

for all $\bar{U}, \bar{V} \in T\bar{M}$, then \bar{M} is called an *almost product Riemannian manifold*.

Next, we denote by $\bar{\nabla}$ the Riemannian connection with respect to g on \bar{M} . We say that \bar{M} is a *locally product Riemannian manifold*, (briefly, *l.p.R. manifold*) if we have

$$(2.3) \quad (\bar{\nabla}_{\bar{U}} F)\bar{V} = 0 ,$$

for all $\bar{U}, \bar{V} \in T\bar{M}$ [22].

2.2. Submanifolds. Let M be a submanifold of a l.p.R. manifold (\bar{M}, g, F) . Let $\bar{\nabla}, \nabla$, and ∇^\perp be the Riemannian, induced Riemannian, and induced normal connection in \bar{M}, M and the normal bundle $T^\perp M$ of M , respectively. Then for all $U, V \in TM$ and $\xi \in T^\perp M$ the Gauss and Weingarten formulas are given by

$$(2.4) \quad \bar{\nabla}_U V = \nabla_U V + h(U, V)$$

and

$$(2.5) \quad \bar{\nabla}_U \xi = -A_\xi U + \nabla_U^\perp \xi$$

where h is the second fundamental form related to shape operator. A corresponding to the normal vector field ξ is given by

$$(2.6) \quad g(h(U, V), \xi) = g(A_\xi U, V) .$$

A submanifold M is said to be *totally geodesic* if its second fundamental form vanishes identically, that is, $h = 0$, or equivalently $A_\xi = 0$. We say that M is *totally umbilical* submanifold in \bar{M} if for all $U, V \in TM$ we have

$$(2.7) \quad h(U, V) = g(U, V)H ,$$

where H is the mean curvature vector field of M in \bar{M} . A normal vector field ξ is said to be parallel, if $\nabla_U^\perp \xi = 0$ for each vector field $U \in TM$.

The Riemannian curvature tensor \bar{R} of \bar{M} is given by

$$(2.8) \quad \bar{R}(\bar{U}, \bar{V}) = [\bar{\nabla}_{\bar{U}}, \bar{\nabla}_{\bar{V}}] - \bar{\nabla}_{[\bar{U}, \bar{V}]},$$

where $\bar{U}, \bar{V} \in T\bar{M}$

Then the Codazzi equation is given by

$$(2.9) \quad (\bar{R}(U, V)W)^\perp = (\bar{\nabla}_U h)(V, W) - (\bar{\nabla}_V h)(U, W)$$

for all $U, V, W \in TM$. Here, \perp denotes the normal component and the covariant derivative of h , denoted by $\bar{\nabla}_U h$ is defined by

$$(2.10) \quad (\bar{\nabla}_U h)(V, W) = \nabla_U^\perp h(V, W) - h(\nabla_U V, W) - h(V, \nabla_U W).$$

Now, we write

$$(2.11) \quad FU = TU + NU,$$

for any $U \in TM$. Here TU is the tangential part of FU , and NU is the normal part of FU . Similarly, for any $\xi \in T^\perp M$, we put

$$(2.12) \quad F\xi = t\xi + \omega\xi,$$

where $t\xi$ is the tangential part of $F\xi$, and $\omega\xi$ is the normal part of $F\xi$.

A distribution \mathcal{D} on a manifold \bar{M} is called *autoparallel* if $\bar{\nabla}_X Y \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$ and called *parallel* if $\bar{\nabla}_U X \in \mathcal{D}$ for any $X \in \mathcal{D}$ and $U \in TM$. If a distribution \mathcal{D} on \bar{M} is autoparallel, then it is clearly integrable, and by Gauss formula \mathcal{D} is totally geodesic in \bar{M} . If \mathcal{D} is parallel then the orthogonal complementary distribution \mathcal{D}^\perp is also parallel, which implies that \mathcal{D} is parallel if and only if \mathcal{D}^\perp is parallel. In this case \bar{M} is locally product of the leaves of \mathcal{D} and \mathcal{D}^\perp . Let M be a submanifold of \bar{M} . For two distributions \mathcal{D}_1 and \mathcal{D}_2 on M , we say that M is $(\mathcal{D}_1, \mathcal{D}_2)$ mixed totally geodesic if for all $X \in \mathcal{D}_1$ and $Y \in \mathcal{D}_2$ we have $h(X, Y) = 0$, where h is the second fundamental form of M [20, 22].

3. HEMI-SLANT SUBMANIFOLDS OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

In this section, we define the notion of hemi-slant submanifold and observe its effect to the tangent bundle of the submanifold and canonical projection operators and start to study hemi-slant submanifolds of a locally product Riemannian manifold.

Let (\bar{M}, g, F) be a locally product Riemannian manifold and let M be a submanifold of \bar{M} . A distribution \mathcal{D} on M is said to be a *slant distribution* if for $X \in \mathcal{D}_p$, the angle θ between FX and \mathcal{D}_p is constant, i.e., independent of $p \in M$ and $X \in \mathcal{D}_p$. The constant angle θ is called the slant angle of the slant distribution \mathcal{D} . A submanifold M of \bar{M} is said to be a *slant submanifold* if the tangent bundle TM of M is slant [12, 17]. Thus, the F -invariant and F -anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold which is neither F -invariant nor F -anti-invariant is called a *proper* slant submanifold.

Definition 3.1. A *hemi-slant submanifold* M of a locally product Riemannian manifold \bar{M} is a submanifold which admits two orthogonal complementary distributions \mathcal{D}^\perp and \mathcal{D}^θ such that

- (a) TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$
- (b) The distribution \mathcal{D}^\perp is F -anti-invariant, i.e., $F\mathcal{D}^\perp \subseteq T^\perp M$.
- (c) The distribution \mathcal{D}^θ is slant with slant angle θ .

In this case, we call θ the slant angle of M . Suppose the dimension of distribution \mathcal{D}^\perp (resp. \mathcal{D}^θ) is p (resp. q). Then we easily see that the following particular cases.

- (d) If $q = 0$, then M is an anti-invariant submanifold [1].
- (e) If $p = 0$ and $\theta = 0$, then M is an invariant submanifold [1].
- (f) If $p = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is a proper slant submanifold [17].
- (g) If $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
- (h) If $p \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold [5].

We say that the hemi-slant submanifold M is *proper* if $p \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

Lemma 3.2. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then we have,*

$$(3.1) \quad F(\mathcal{D}^\perp) \perp N(\mathcal{D}^\theta) .$$

Proof. For any $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$, using (2.2) and (2.11), we have $g(FX, NZ) = g(FX, FZ) = g(X, Z) = 0$. This completes the proof. \square

In view of Lemma 3.2, for a hemi-slant submanifold M of a l.p.R. manifold \bar{M} , the normal bundle $T^\perp M$ of M is decomposed as

$$(3.2) \quad T^\perp M = F(\mathcal{D}^\perp) \oplus N(\mathcal{D}^\theta) \oplus \mu ,$$

where μ is the orthogonal complementary distribution of $F(\mathcal{D}^\perp) \oplus N(\mathcal{D}^\theta)$ in $T^\perp M$ and it is invariant subbundle of $T^\perp M$ with respect to F .

The following facts follow easily from (2.1), (2.11) and (2.12) and will be used later.

$$(3.3) \quad \begin{aligned} (a) \quad T^2 + tN &= I, & (b) \quad \omega^2 + Nt &= I, \\ (c) \quad NT + \omega N &= 0, & (d) \quad Tt + t\omega &= 0. \end{aligned}$$

As in a slant submanifold [17], for a hemi-slant submanifold M of a l.p.R. manifold \bar{M} , we have

$$(3.4) \quad T^2 Z = \cos^2 \theta Z ,$$

$$(3.5) \quad g(TZ, TW) = \cos^2 \theta g(Z, W)$$

and

$$(3.6) \quad g(NZ, NW) = \sin^2 \theta g(Z, W) ,$$

where $Z, W \in \mathcal{D}^\theta$.

Lemma 3.3. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then we have,*

$$(3.7) \quad (a) \quad T(\mathcal{D}^\perp) = \{0\}, \quad (b) \quad T(\mathcal{D}^\theta) = \mathcal{D}^\theta .$$

Proof. Since \mathcal{D}^\perp is anti-invariant with respect to F , (a) follows from (2.11). For any $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^\perp$, using (2.1), (2.2) and (2.11), we have $g(TZ, X) = g(FZ, X) = g(Z, FX) = 0$. Hence, we conclude that $T(\mathcal{D}^\theta) \perp \mathcal{D}^\perp$. Since $T(\mathcal{D}^\theta) \subseteq TM$, it follows that $T(\mathcal{D}^\theta) \subseteq \mathcal{D}^\theta$. Let W be in \mathcal{D}^θ . Then using (3.4), we have

$W = \frac{1}{\cos^2\theta}(\cos^2\theta W) = \frac{1}{\cos^2\theta}T^2W = \frac{1}{\cos^2\theta}T(TW)$. So, we find $W \in T(\mathcal{D}^\theta)$. It follows that $\mathcal{D}^\theta \subseteq T(\mathcal{D}^\theta)$. Thus, we get the assertion (b). \square

Thanks to Theorem 3.1 [17], we characterize hemi-slant submanifolds of a l.p.R. manifold.

Theorem 3.4. *Let M be a submanifold of a l.p.R. manifold \bar{M} . Then M is a hemi-slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ and a distribution \mathcal{D} on M such that*

- (a) $\mathcal{D} = \{U \in TM \mid T^2U = \lambda U\}$,
- (b) for any $X \in TM$ orthogonal to \mathcal{D} , $TX = 0$.

Moreover, in this case $\lambda = \cos^2\theta$, where θ is the slant angle of M .

Proof. Let M be a hemi-slant submanifold of \bar{M} . By the definition of hemi-slant submanifold, we have $\mathcal{D} = \mathcal{D}^\theta$ and $\lambda = \cos^2\theta$. So, (a) follows. (b) follows from Lemma 3.3. Conversely, (a) and (b) imply $TM = \mathcal{D}^\perp \oplus \mathcal{D}$. Since $T(\mathcal{D}) \subseteq \mathcal{D}$, we conclude that \mathcal{D}^\perp is an anti-invariant distribution from (b). \square

Example. Consider the Euclidean 6-space \mathbb{R}^6 with usual metric g . Define the almost product structure F on (\mathbb{R}^6, g) by

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad F\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \quad i = 1, 2, 3.$$

Where $(x_1, x_2, x_3, y_1, y_2, y_3)$ are natural coordinates of \mathbb{R}^6 . Then $\bar{M} = (\mathbb{R}^6, g, F)$ be an almost product Riemannian manifold. Furthermore, it is easy to see that \bar{M} is a l.p.R. manifold. Let M be a submanifold of \bar{M} defined by

$$f(u, v, w) = \left(\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}, u + v, \frac{w}{\sqrt{2}}, \frac{w}{\sqrt{2}}, 0\right), \quad u \neq 0.$$

Then, a local frame of TM is given by

$$\begin{aligned} X &= \frac{\partial}{\partial x_3}, \\ Z &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \\ W &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2}. \end{aligned}$$

By using the almost product structure F above, we see that FX is orthogonal to TM , thus $\mathcal{D}^\perp = \text{span}\{X\}$. Moreover, it is not difficult to see that $\mathcal{D}^\theta = \text{span}\{Z, W\}$ is a slant distribution with slant angle $\theta = \pi/3$. Thus, M is a proper hemi-slant submanifold of \bar{M} .

4. INTEGRABILITY

In this section, we give a necessary and sufficient condition for the integrability of the slant distribution of the hemi-slant submanifold. After that we prove that the anti invariant distribution of the hemi-slant submanifold is always integrable and give some applications of this result.

Let M be a submanifold of a l.p.R. manifold \bar{M} . For any $U, V \in TM$, we have $\bar{\nabla}_U FV = F\bar{\nabla}_U V$ from (2.3). Then, using (2.4-2.5), (2.11-2.12) and identifying the components from TM and $T^\perp M$, we have the following.

Lemma 4.1. *Let M be a submanifold of a l.p.R. manifold \bar{M} . Then we have,*

$$(4.1) \quad \nabla_U TV - A_{NV}U = T\nabla_U V + t h(U, V),$$

$$(4.2) \quad h(U, TV) + \nabla_U^\perp NV = N\nabla_U V + \omega h(U, V) .$$

for all $U, V \in TM$.

In a similar way, we have that:

Lemma 4.2. *Let M be a submanifold of a l.p.R. manifold \bar{M} . Then we have,*

$$(4.3) \quad \nabla_U t\xi - A_{\omega\xi}U = -TA_\xi U + t\nabla_U^\perp \xi ,$$

$$(4.4) \quad h(U, t\xi) + \nabla_U^\perp \omega\xi = -NA_\xi U + \omega\nabla_U^\perp \xi$$

for any $U \in TM$ and $\xi \in T^\perp M$.

Theorem 4.3. *Let M be a hemi-slant manifold of a l.p.R. manifold \bar{M} . Then, the slant distribution \mathcal{D}^θ is integrable if and only if*

$$(4.5) \quad A_{NZ}W - A_{NW}Z + \nabla_Z TW - \nabla_W TZ \in \mathcal{D}^\theta$$

for any $Z, W \in \mathcal{D}^\theta$.

Proof. From (4.1), we have

$$(4.6) \quad \nabla_Z TW - A_{NW}Z = T\nabla_Z W + t h(Z, W)$$

and

$$(4.7) \quad \nabla_W TZ - A_{NZ}W = T\nabla_W Z + t h(W, Z)$$

for any $Z, W \in \mathcal{D}^\theta$. Since h is a symmetric $(0, 2)$ -type tensor field, from (4.6) and (4.7), we get

$$(4.8) \quad A_{NZ}W - A_{NW}Z + \nabla_Z TW - \nabla_W TZ = T[Z, W] .$$

Thus, our assertion follows from (3.7-b) and (4.8). \square

The following we give an application of Theorem 4.3.

Theorem 4.4. *Let M be a hemi-slant manifold of a l.p.R. manifold \bar{M} . If M is \mathcal{D}^θ -totally geodesic, then the slant distribution \mathcal{D}^θ is integrable.*

Proof. Suppose that M is \mathcal{D}^θ -totally geodesic, that is, for any $Z, W \in \mathcal{D}^\theta$ we have

$$(4.9) \quad h(Z, W) = 0.$$

Thus, from (4.1), using (4.9), we have

$$(4.10) \quad A_{NZ}W - \nabla_W TZ = -T\nabla_W Z$$

and similarly

$$(4.11) \quad A_{NW}Z - \nabla_Z TW = -T\nabla_Z W .$$

From (4.10) and (4.11), using Lemma 3.3, we get

$$(4.12) \quad g(A_{NZ}W - A_{NW}Z + \nabla_Z TW - \nabla_W TZ, X) = g(T[Z, W], X) = 0$$

for any $X \in \mathcal{D}^\perp$. The last equation (4.12) says that

$$A_{NZ}W - A_{NW}Z + \nabla_Z TW - \nabla_W TZ \in \mathcal{D}^\theta$$

and by Theorem 4.3, we deduce that \mathcal{D}^θ is integrable. \square

Lemma 4.5. *Let M be a hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then,*

$$(4.13) \quad A_{NX}Y = -A_{NY}X$$

for any $X, Y \in \mathcal{D}^\perp$.

Proof. For any $X \in \mathcal{D}^\perp$ and $U \in TM$, using (3.7-a), we have

$$(4.14) \quad -T\nabla_U X = A_{NX}U + th(U, X)$$

from (4.1). Let Y be in \mathcal{D}^\perp . Using (3.7-b), we obtain

$$(4.15) \quad 0 = -g(T\nabla_U X, Y) = g(A_{NX}U, Y) + g(th(U, X), Y)$$

from (4.14). On the other hand, using (2.2), (2.6), (2.11) and (2.12), we find

$$(4.16) \quad g(th(U, X), Y) = g(A_{NY}U, X).$$

Thus, from (4.15) and (4.16), we deduce that

$$(4.17) \quad g(A_{NX}Y + A_{NY}X, U) = 0.$$

This equation gives (4.13). \square

Theorem 4.6. *Let M be a hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then the anti-invariant distribution \mathcal{D}^\perp is integrable if and only if*

$$(4.18) \quad A_{NX}Y = A_{NY}X$$

for all $X, Y \in \mathcal{D}^\perp$.

Proof. From (4.1), using (3.7-a), we have

$$(4.19) \quad -A_{NY}X = T\nabla_X Y + th(X, Y)$$

for all $X \in \mathcal{D}^\perp$. By interchanging X and Y in (4.19), then subtracting it from (4.19) we obtain

$$(4.20) \quad A_{NX}Y - A_{NY}X = T[X, Y].$$

Because of (3.7-a), we know that \mathcal{D}^\perp is integrable if and only if $T[X, Y] = 0$ for all $X, Y \in \mathcal{D}^\perp$. So, our assertion comes from (4.20). \square

By Lemma 4.5 and Theorem 4.6, we have the following result.

Corollary 4.7. *Let M be a hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then the anti-invariant distribution \mathcal{D}^\perp is integrable if and only if*

$$(4.21) \quad A_{NX}Y = 0$$

for all $X, Y \in \mathcal{D}^\perp$.

Now, we give main result of this section.

Theorem 4.8. *Let M be a hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then the anti-invariant distribution \mathcal{D}^\perp is always integrable.*

Proof. Let \bar{M} be a l.p.R. manifold with Riemannian metric g and almost product structure F . Define the symmetric (0,2)-type tensor field Ω by $\Omega(\bar{U}, \bar{V}) = g(F\bar{U}, \bar{V})$ on the tangent bundle $T\bar{M}$. It is not difficult to see that $(\nabla_{\bar{U}}\Omega)(\bar{V}, \bar{W}) = g((\nabla_{\bar{U}}F)\bar{V}, \bar{W})$ on $T\bar{M}$. Thus, because of (2.3), we deduce that

$$3 d\Omega(\bar{V}, \bar{W}, \bar{U}) = \mathcal{G}(\nabla_{\bar{U}}\Omega)(\bar{V}, \bar{W}) = 0$$

for all $\bar{U}, \bar{V}, \bar{W} \in T\bar{M}$, that is, $d\Omega \equiv 0$, where \mathcal{G} denotes the cyclic sum over $\bar{U}, \bar{V}, \bar{W} \in T\bar{M}$. Next, for any $X, Y \in \mathcal{D}^\perp$ and $U \in TM$ we have

$$\begin{aligned} 0 &= 3 d\Omega(U, X, Y) = U \Omega(X, Y) + X \Omega(Y, U) + Y \Omega(U, X) \\ &\quad - \Omega([U, X], Y) - \Omega([X, Y], U) - \Omega([Y, U], X) \\ &= g(T[Y, X], U). \end{aligned}$$

It follows that $T[X, Y] = 0$ and because of (3.7-a), $[Y, X] \in \mathcal{D}^\perp$. \square

We remark that we used Tripathi's technique [8] in the proof above.

Corollary 4.9. *Let M be a hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then the following facts hold:*

$$(4.22) \quad A_{ND^\perp} D^\perp = 0$$

$$(4.23) \quad A_{NX} Z \in D^\theta, \quad \text{i.e., } A_{ND^\perp} D^\theta \subseteq D^\theta$$

and

$$(4.24) \quad g(h(TM, \mathcal{D}^\perp), N\mathcal{D}^\perp) = 0,$$

where $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

Proof. (4.22) follows from Corollary 4.7 and Theorem 4.8. (4.23) follows from (4.22). Finally, using (2.6), (4.22) gives (4.24). \square

Next, we give another application of Theorem 4.8.

Theorem 4.10. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . The anti-invariant distribution \mathcal{D}^\perp defines a totally geodesic foliation on M if and only if $h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp N\mathcal{D}^\theta$.*

Proof. For $X, Y \in \mathcal{D}^\perp$, we put $\nabla_X Y = {}^\perp\nabla_X Y + {}^\theta\nabla_X Y$, where ${}^\perp\nabla_X Y$ (resp. ${}^\theta\nabla_X Y$) denotes the anti-invariant (resp. slant) part of $\nabla_X Y$. Then using Lemma 3.3 and (3.5), for any $Z \in \mathcal{D}^\theta$ we have

$$(4.25) \quad g(\nabla_X Y, Z) = g({}^\theta\nabla_X Y, Z) = \frac{1}{\cos^2\theta} g(T{}^\theta\nabla_X Y, TZ) = \frac{1}{\cos^2\theta} g(T\nabla_X Y, TZ).$$

On the other hand, from (4.1), we have

$$(4.26) \quad T\nabla_X Y + t h(X, Y) = -A_{NY} X = 0,$$

since the distribution \mathcal{D}^\perp is integrable. So, using (4.26), from (4.25), we get

$$(4.27) \quad g(\nabla_X Y, Z) = -\frac{1}{\cos^2\theta} g(t h(X, Y), TZ) = -\frac{1}{\cos^2\theta} g(Fh(X, Y), TZ).$$

Here, using (2.2), (2.11) and (3.4), we find

$$(4.28) \quad g(Fh(X, Y), TZ) = g(h(X, Y), NTZ).$$

From (4.27) and (4.28), we get

$$(4.29) \quad g(\nabla_X Y, Z) = -\frac{1}{\cos^2\theta} g(h(X, Y), NTZ).$$

Since $TZ \in \mathcal{D}^\theta$, our assertion comes from (4.29). \square

5. HEMI-SLANT PRODUCT

In this section, we give a necessary and sufficient condition for a proper hemi-slant submanifold to be a hemi-slant product.

Definition 5.1. A proper hemi-slant submanifold M of a l.p.R. manifold \bar{M} is called a hemi-slant product if it is locally product Riemannian of an anti-invariant submanifold M_\perp and a proper slant submanifold M_θ of \bar{M} .

Now, we are going to examine the problem when a proper hemi-slant submanifold of a l.p.R. manifold is a hemi-slant product?

We first give a result which is equivalent to Theorem 4.10.

Theorem 5.2. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then the anti-invariant \mathcal{D}^\perp defines a totally geodesic foliation on M if and only if*

$$(5.1) \quad g(A_{NY}Z, X) = -g(A_{NZ}Y, X),$$

where $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

Proof. For any $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$, using (2.4), (2.2), and (2.3), we have

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X FY, FZ).$$

Hence, using (2.11), (2.4), (2.5) and (2.2), we obtain

$$g(\nabla_X Y, Z) = -g(A_{NY}X, TZ) + g(\nabla_X Y, FNZ) + g(h(X, Y), FNZ).$$

Here, using (3.3)-c, (3.3)-a, (2.12) and (3.4), we have

$FNZ = tNZ - NTZ$ and $tNZ = Z - T^2Z = \sin^2\theta Z$. Thus, with the help of (2.6), we get

$$g(\nabla_X Y, Z) = -g(A_{NY}X, TZ) + \sin^2\theta g(\nabla_X Y, Z) - g(A_{NTZ}Y, X).$$

After some calculations, we find

$$\cos^2\theta g(\nabla_X Y, Z) = -g(A_{NY}TZ, X) - g(A_{NTZ}Y, X).$$

It follows that the distribution \mathcal{D}^\perp defines a totally geodesic foliation on M if and only if

$$(5.2) \quad g(A_{NY}TZ, X) = -g(A_{NTZ}Y, X).$$

Putting $Z = TZ$ in (5.2), we obtain (5.1) and vice versa. \square

Theorem 5.3. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then the distribution \mathcal{D}^θ defines a totally geodesic foliation on M if and only if*

$$(5.3) \quad g(A_{NX}W, Z) = -g(A_{NW}X, Z),$$

where $X, Y \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$.

Proof. Using (2.4), (2.2), and (2.3), we have $g(\nabla_Z W, X) = g(\bar{\nabla}_Z FW, FX)$ for any $Z, W \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^\perp$. Next, using (2.11) and (3.1), obtain $g(\nabla_Z W, X) = -g(TW, \bar{\nabla}_Z NX) - g(NW, \bar{\nabla}_Z FX)$. Hence, using (2.5) and (2.1), we get $g(\nabla_Z W, X) = g(TW, A_{NX}Z) - g(FNW, \bar{\nabla}_Z X)$. With the help of (2.12), (3.3)-(a), (3.3)-(c) and (2.4), we arrive at

$$g(\nabla_Z W, X) = -g(A_{NX}Z, TW) - \sin^2\theta g(\nabla_Z X, W) + g(h(X, Z), NTW).$$

Upon direct calculation, we find

$$\cos^2\theta \, g(\nabla_Z W, X) = g(A_{NX}TW, Z) + g(A_{NTW}X, Z)$$

So, we deduce that the slant distribution \mathcal{D}^θ defines a totally geodesic foliation if and only if

$$(5.4) \quad g(A_{NX}TW, Z) = -g(A_{NTW}X, Z),$$

By putting $W = TW$, we see that the last equation is equivalent to the equation (5.3). \square

Thus, from Theorems 5.2 and 5.3, we obtain the expected result.

Corollary 5.4. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . Then M is a hemi-slant product manifold $M = M_\perp \times M_\theta$ if and only if*

$$(5.5) \quad A_{NX}Z = -A_{NZ}X,$$

where $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

6. HEMI-SLANT SUBMANIFOLDS WITH PARALLEL CANONICAL STRUCTURES

In this section, we get several results for the hemi-slant submanifolds with parallel canonical structures using the previous results.

Let M be any submanifold of a l.p.R. manifold \bar{M} with the endomorphism T and the normal bundle valued 1-form N defined by (2.11). We put

$$(6.1) \quad (\bar{\nabla}_U T)V = \nabla_U TV - T\nabla_U V$$

and

$$(6.2) \quad (\bar{\nabla}_U N)V = \nabla_U^\perp NV - N\nabla_U V$$

for any $U, V \in TM$. Then the endomorphism T (resp. 1-form N) is parallel if $\bar{\nabla}T \equiv 0$ (resp. $\bar{\nabla}N \equiv 0$). From (4.1) and (4.2) we have

$$(6.3) \quad (\bar{\nabla}_U T)V = A_{NV}U + t h(U, V)$$

and

$$(6.4) \quad (\bar{\nabla}_U N)V = \omega h(U, V) - h(U, TV),$$

respectively.

Theorem 6.1. *Let M be any submanifold of a l.p.R. manifold \bar{M} . Then T is parallel, i.e., $\bar{\nabla}T \equiv 0$ if and only if*

$$(6.5) \quad A_{NV}U = -A_{NU}V,$$

for all $U, V \in TM$.

Proof. For any $U, V, W \in TM$ from (6.3), we have

$$g((\bar{\nabla}_W T)V, U) = g(A_{NV}W, U) + g(th(W, V), U).$$

Hence, using (2.12), (2.2) and (2.11), we obtain

$$g((\bar{\nabla}_W T)V, U) = g(A_{NV}W, U) + g(h(W, V), NU).$$

Since A is self-adjoint, with the help of (2.6), we get

$$(6.6) \quad g((\bar{\nabla}_W T)V, U) = g(A_{NV}U, W) + g(A_{NU}V, W).$$

Thus, our assertion comes from (6.6). \square

Theorem 6.2. *Let M be a proper hemi-slant submanifold of a l.p.R. manifold \bar{M} . If T is parallel, then M is a hemi-slant product. The converse is true, if $h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp N\mathcal{D}^\theta$.*

Proof. Let X be in \mathcal{D}^\perp and Z in \mathcal{D}^θ . If T is parallel, then from (6.5), we have

$$(6.7) \quad A_{NX}Z = -A_{NZ}X.$$

Thus, by Corollary 5.4, we conclude that M is a hemi-slant product. Conversely, if M is a hemi-slant product and $h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp N\mathcal{D}^\theta$, then for any Z, W and $V \in \mathcal{D}^\theta$, we have $g(A_{NZ}W, V) = g(h(V, W), NZ) = 0$. It means that $A_{NZ}W \in \mathcal{D}^\perp$. Now, let calculate $g(A_{NZ}W, X)$ for $X \in \mathcal{D}^\perp$. Since M is a hemi-slant product and A is self-adjoint $g(A_{NZ}W, X) = g(A_{NZ}X, W) = -g(A_{NX}Z, W) = -g(A_{NX}W, Z) = -g(A_{NW}X, Z) = -g(A_{NW}Z, X)$.

Hence, we deduce

$$(6.8) \quad A_{NZ}W = -A_{NW}Z,$$

for all $Z, W \in \mathcal{D}^\theta$.

Thus, from (4.13), (6.7) and (6.8), we obtain (6.5) and by Theorem 6.1, T is parallel. \square

Theorem 6.3. *Let M be a proper hemi-slant submanifold of \bar{M} . If N is parallel, then*

- (a) $A_\mu \mathcal{D}^\perp = 0$, (b) $A_{N\mathcal{D}^\theta} \mathcal{D}^\perp = 0$, (c) $A_{N\mathcal{D}^\perp} \mathcal{D}^\theta = 0$,
 (d) M is a hemi-slant product, (e) M is $(\mathcal{D}^\perp, \mathcal{D}^\theta)$ -mixed totally geodesic.

Proof. Let N be parallel, it follows from (6.4) that

$$(6.9) \quad h(U, TV) = \omega h(U, V)$$

for any $U, V \in TM$. Then, for any $X \in \mathcal{D}^\perp$, we have

$$(6.10) \quad \omega h(U, X) = 0$$

from (6.9). For any $\xi \in \mu$, using (2.11), (2.2) and (2.6), we have

$$g(\omega h(U, X), \xi) = g(h(U, X), F\xi) = g(A_{F\xi}X, U).$$

Thus, using (6.10) we get

$$(6.11) \quad g(A_{F\xi}X, U) = 0.$$

Since μ is invariant with respect to F , the assertion (a) comes from (6.11). Now, take $Z \in \mathcal{D}^\theta$, after some calculations, we find

$$g(A_{NZ}X, U) = g(\omega h(U, X), NZ).$$

So, using (6.10), we get $g(A_{NZ}X, U) = 0$, which is equivalent to the assertion (b). On the other hand, for any $X \in \mathcal{D}^\perp$, using (2.2), (2.11), (2.12) and (6.9), we have

$$\begin{aligned} 0 &= g(h(U, Z), X) = g(Fh(U, Z), FX) = g(\omega h(U, Z), FX) \\ &= g(h(U, TZ), FX) = g(h(U, TZ), NX), \end{aligned}$$

that is, $g(h(U, TZ), NX) = 0$. Putting $Z = TZ$ in last equation, we obtain

$$\cos^2 \theta g(h(U, Z), NX) = \cos^2 \theta g(A_{NX}Z, U) = 0.$$

Since $\theta \neq \frac{\pi}{2}$, the assertion (c) follows. The assertion (d) follows from the assertions (b), (c) and (5.5). Lastly, using (3.4), from (6.9), we have

$\omega^2 h(X, Z) = \omega h(X, TZ) = h(X, T^2 Z) = \cos^2 \theta h(X, Z)$. On the other hand, using (3.7)-(a), we have $\omega^2 h(X, Z) = \omega^2 h(Z, X) = \omega h(Z, TX) = 0$. Thus, we get $\cos^2 \theta h(X, Z) = 0$. Since $\theta \neq \frac{\pi}{2}$, we deduce that $h(X, Z) = 0$, which proves that the last assertion. \square

7. TOTALLY UMBILICAL HEMI-SLANT SUBMANIFOLDS

In this section we shall give two characterization theorems for the totally umbilical proper hemi-slant submanifolds of a l.p.R. manifold. First we prove

Theorem 7.1. *If M is a totally umbilical proper hemi-slant submanifold of a l.p.R. manifold \bar{M} , then either the anti-invariant distribution \mathcal{D}^\perp is 1-dimensional or the mean curvature vector field H of M is perpendicular to $F(\mathcal{D}^\perp)$. Moreover, if M is a hemi-slant product, then $H \in \mu$.*

Proof. Since M is a totally umbilical proper hemi-slant submanifold either $\dim(\mathcal{D}^\perp) = 1$ or $\dim(\mathcal{D}^\perp) > 1$. If $\dim(\mathcal{D}^\perp) = 1$, it is obvious. If $\dim(\mathcal{D}^\perp) > 1$, then we can choose $X, Y \in \mathcal{D}^\perp$ such that $\{X, Y\}$ is orthonormal. By using (2.11), (2.7), (2.6) and (4.22), we have

$$(7.1) \quad g(H, FY) = g(h(X, X), NY) = g(A_{NY}X, X) = 0$$

It means that

$$(7.2) \quad H \perp F(\mathcal{D}^\perp).$$

Moreover, if M is a hemi-slant product, for any $Z \in \mathcal{D}^\theta$, using (5.5) and (2.7), we have

$$\begin{aligned} g(H, NZ) &= g(h(X, X), NZ) = g(A_{NZ}X, X) = -g(A_{NX}Z, X) \\ &= -g(h(Z, X), NX) = 0. \end{aligned}$$

Hence, it follows that

$$(7.3) \quad H \perp N(\mathcal{D}^\theta).$$

Thus, using (7.2) and (7.3) from (3.2), we get $H \in \mu$. \square

Before giving the second result of this section, recall that the following fact about locally product Riemannian manifolds.

Let $M_1(c_1)$ (resp. $M_2(c_2)$) be a real space form with sectional curvature c_1 (resp. c_2). Then the Riemannian curvature tensor \bar{R} of the locally product Riemannian manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ has the form

$$(7.4) \quad \begin{aligned} \bar{R}(\bar{U}, \bar{V})\bar{W} &= \frac{1}{4}(c_1 + c_2) \left\{ g(\bar{V}, \bar{W})\bar{U} - g(\bar{U}, \bar{W})\bar{V} + g(F\bar{V}, \bar{W})F\bar{U} - g(F\bar{U}, \bar{W})F\bar{V} \right\} \\ &\quad + \frac{1}{4}(c_1 - c_2) \left\{ g(F\bar{V}, \bar{W})\bar{U} - g(F\bar{U}, \bar{W})\bar{V} + g(\bar{V}, \bar{W})F\bar{U} - g(\bar{U}, \bar{W})F\bar{V} \right\}, \end{aligned}$$

where $\bar{U}, \bar{V}, \bar{W} \in T\bar{M}$ [22].

Theorem 7.2. *Let M be a totally umbilical hemi-slant submanifold with parallel mean curvature vector field H of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ with $c_1 \neq c_2$. Then, M can not be proper.*

Proof. Let $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$ be two unit vector fields. Since H is parallel, using (2.10) and (2.7) from the Codazzi equation (2.9), we have

$$(7.5) \quad (\overline{R}(X, Z)X)^\perp = -\nabla_Z^\perp H = 0.$$

On the other hand, the equation (7.4) gives

$$(7.6) \quad \overline{R}(X, Z)X = -\frac{1}{4} \left\{ (c_1 + c_2)Z + (c_1 - c_2)FZ \right\}.$$

Taking the normal component of (7.6), we get

$$(7.7) \quad (\overline{R}(X, Z)X)^\perp = -\frac{1}{4}(c_1 - c_2)NZ,$$

which contradicts (7.5). \square

We have immediately from Theorem 7.2. that:

Corollary 7.3. *There exists no totally geodesic proper hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ with $c_1 \neq c_2$.*

8. RICCI CURVATURE OF HEMI-SLANT SUBMANIFOLDS

In this section, we obtain a basic inequality involving Ricci curvature and the squared mean curvature of a hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. We first represent the following fundamental facts about this topic.

Let \bar{M} be a n -dimensional Riemannian manifold equipped with a Riemannian metric g and $\{e_1, \dots, e_n\}$ be an orthonormal basis for $T_p\bar{M}$, $p \in \bar{M}$. Then the *Ricci tensor* \overline{S} is defined by

$$(8.1) \quad \overline{S}(U, V) = \sum_{i=1}^n \overline{R}(e_i, U, V, e_i),$$

where $U, V \in T_p\bar{M}$. For a fixed $i \in \{1, \dots, n\}$, the *Ricci curvature* of e_i , denoted by $\overline{Ric}(e_i)$, is given by

$$(8.2) \quad \overline{Ric}(e_i) = \sum_{i \neq j}^n \overline{K}_{ij},$$

where $\overline{K}_{ij} = g(\overline{R}(e_i, e_j)e_j, e_i)$ is the *sectional curvature* of the plane spanned by the plane spanned by e_i and e_j at $p \in \bar{M}$. Let Π_k be a k -plane of $T_p\bar{M}$ and $\{e_1, \dots, e_k\}$ any orthonormal basis of Π_k . For a fixed $i \in \{1, \dots, k\}$, the *k-Ricci curvature* [9] of Π_k at e_i , denoted by $\overline{Ric}_{\Pi_k}(e_i)$, is defined by

$$(8.3) \quad \overline{Ric}_{\Pi_k}(e_i) = \sum_{i \neq j}^k \overline{K}_{ij}.$$

It is easy to see that $\overline{Ric}_{(T_p\bar{M})}(e_i) = \overline{Ric}(e_i)$ for $1 \leq i \leq n$, since $\Pi_n = T_p\bar{M}$.

We now recall that the following basic inequality [10, Theorem 3.1] involving Ricci curvature and the squared mean curvature of a submanifold of a Riemannian manifold.

Theorem 8.1. ([10, Theorem 3.1]) *Let M be an m -dimensional submanifold of a Riemannian manifold \bar{M} . Then, for any unit vector $X \in T_p M$, we have*

$$(8.4) \quad Ric(X) \leq \frac{1}{4}m^2\|H\|^2 + \bar{Ric}_{(T_p M)}(X)$$

where $Ric(X)$ is the Ricci curvature of X .

Of course, the equality case of (8.4) was also discussed in [10], but we will not deal with the equality case in this paper.

Now, we are ready to state main result of this section.

Theorem 8.2. *Let M be an m -dimensional hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. Then, for unit vector $V \in T_p M$, we have*

$$(8.5) \quad 4Ric(V) \leq m^2\|H\|^2 + (c_1 + c_2) \left\{ (m-1) + \sum_{i=2}^m g(Te_i, e_i)g(TV, V) - \|TV\|^2 + g(TV, V) \right\} + (c_1 - c_2) \left\{ \sum_{i=2}^m g(Te_i, e_i) + (m-1)g(TV, V) \right\}$$

where $\{V, e_2, \dots, e_m\}$ is an orthonormal basis for $T_p M$.

Proof. Let M be an m -dimensional hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. Then for any unit vector $V \in T_p M$, using (7.4) and (2.11) from (8.3) we have

$$(8.6) \quad 4\bar{Ric}_{(T_p M)}(V) = (c_1 + c_2) \left\{ (m-1) + \sum_{i=2}^m g(Te_i, e_i)g(TV, V) - \|TV\|^2 + g(TV, V) \right\} + (c_1 - c_2) \left\{ \sum_{i=2}^m g(Te_i, e_i) + (m-1)g(TV, V) \right\}$$

Thus, using (8.6) in (8.4) we get (8.5). \square

Remark 8.3. In general, $g(F\bar{V}, \bar{V}) \neq 0$ for any unit vector $\bar{V} \in T_p \bar{M}$ in a l.p.R. manifold \bar{M} , contrary to almost Hermitian ($g(J\bar{V}, \bar{V}) = 0$) and almost contact ($(g(\varphi\bar{V}, \bar{V}) = 0)$ manifolds. However, we can establish that the almost product structure F in a l.p.R. manifold \bar{M} such that $g(F\bar{V}, \bar{V}) = 0$, for all $\bar{V} \in T_p \bar{M}$. In fact, if \bar{M} is an even dimensional l.p.R. manifold with an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$, then we can define F by

$$F(e_j) = e_{n+j}, \quad F(e_{n+j}) = e_j, \quad j \in \{1, 2, \dots, n\}.$$

Hence, we observe easily that the almost product structure F satisfies

$$(8.7) \quad g(Fe_j, e_j) = 0.$$

For example, the almost product structure F in example of section 3, satisfies the condition (8.7). On the other hand, because of Lemma 3.3 and the equation (3.5), we have $TV = 0$, if $V \in \mathcal{D}^\perp$ and $\|TV\|^2 = \cos^2\theta$, if $V \in \mathcal{D}^\theta$ and $\|V\| = 1$, respectively. Thus, by Theorem 8.2 we get the following two results.

Corollary 8.4. *Let M be an m -dimensional anti-invariant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. If the almost product structure F of \bar{M} satisfies the condition (8.7), then we have*

$$4\text{Ric}(V) \leq m^2\|H\|^2 + (c_1 + c_2)(m - 1),$$

where $V \in T_p M$ is any unit vector.

Corollary 8.5. *Let M be an m -dimensional slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. If the almost product structure F of \bar{M} satisfies the condition (8.7), then we have*

$$4\text{Ric}(Z) \leq m^2\|H\|^2 + (c_1 + c_2)\{(m - 1) - \cos^2\theta\},$$

where $Z \in T_p M$ is any unit vector.

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